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Polynomial Interpolation with Prescribed Analytic Functionals

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We study the reconstruction of an analytic function of several complex variables by means of interpolating polynomials obtained from pieces of information given by functionals of derivatives of the function. Several classical interpolation methods are examples of our general problem. © 1993 Academic Press, Inc.

INTRODUCTION

Let us suppose that, by some process, you know m numerical pieces of information on a function f; let us suppose also that you can construct a polynomial (of degree less than or equal to $m-1$) which, by the same process, gives the same m informations: you have found an interpolating polynomial (for the process in question) of the function f.

When the m pieces of information are the values of f at m distinct points, the interpolating polynomial is only the Lagrange polynomial; when these are the values of the m first derivatives of f at the point x, then the polynomial is the Talyor's expansion of f at order m and at the point x.

The usual problem is: if the number of pieces of information grows larger and larger, does the interpolating polynomial converge (uniformly) to the function f?

In general no, but sometimes yes (as is well known for the above examples) when f has appropriate analytic properties.

We study such a problem for functions of several complex variables and a quite general process: the information is given by analytic functionals of the derivatives of the function f; see Problem 1.1.

This work finds its origin essentially in the study by Gelfond [11] of the general divided differences interpolation which already generalized previous work of Gontcharoff; see [12]. In the multivariate context this procedure has been studied by Cavaretta *et al.* for the definition, see [8], and by Goodman and Sharma for the convergence; see [13]. The methods of

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Gelfond, Gontcharoff, Goodman, and Sharma may be followed quite closely in our more general problem (except in Section 4).

Let us finally specify some notation. If Ω is an open subset of $\mathbb{C}^n$, then $H(\Omega)$ denotes the space of analytic functions on Ω endowed with the topology of uniform convergence on compact subset of Ω; $H'(\Omega)$ is the space of continuous linear forms on $H(\Omega)$, whose elements are usually called analytic functionals. If μ is an analytic functional and f a function depending on ξ, ζ, then $\mu_\xi(f)$ means $\mu(f(\cdot, \zeta))$. Sometimes we will also write (incorrectly) $\mu(f(z))$ for $\mu(f)$, for example $\mu(z^2)$ will mean $\mu(z \to z^2)$.

$C(a, b, ...)$ denotes a constant depending on $a, b, ...$, but not always with the same value.

1. Interpolating Polynomials

Problem 1.1. Let Ω be an open subset of $\mathbb{C}^n$, $\alpha^i \in H'(\Omega)$, $i = 0 \cdots d$, $f \in H(\Omega)$, find a polynomial $p(z)$ of degree $\leqslant d$ such that

$$\alpha^i(D^\beta f) = \alpha^i(D^\beta p), \qquad |\beta| = i \quad \text{and} \quad 0 \leqslant i \leqslant d, \tag{1}$$

where $\beta = (\beta_1, ..., \beta_n)$, $|\beta| = \sum \beta_i$, $D^\beta = \partial^{|\beta|}/\partial z_1^{\beta_1} \cdots \partial z_n^{\beta_n}$. Further, if p exists does $p(z)$ converge to $f(z)$ when the number of α^i becomes arbitrarily large?

Of course a positive answer will require serious hypothesis on f and on the functionals.

In order that the polynomial $p(z)$ exists uniquely for each $f \in H(\Omega)$ it is necessary and sufficient (as is easily seen) that $\alpha^i(1) \neq 0$, $i = 0, ..., d$, without diminishing the generality of Problem 1.1 we may suppose that $\alpha^i(1) = 1$, $i = 0, ..., d$. In this condition the polynomial $p(z)$, which we will denote by $L(\alpha, f, z)$, may be written in the form

$$L(\alpha, f, z) = \sum_{|\beta| \leqslant d} \alpha^{|\beta|}(D^\beta f)\, Q_\beta(z), \tag{2}$$

where the polynomials Q_β (which we call basis polynomials for α) are defined by the following inductive relation:

$$Q_0(z) = 1. \tag{3}$$

If $\beta = (0, ..., 0, 1, 0, ..., 0)$ with the "1" at the ith place, then

$$Q_\beta(z) = z_i - \alpha^0(z_i). \tag{4}$$

If Q_β are constructed for $|\beta| \leqslant k-1$ and if $|\gamma| = k$ then we define

$$\gamma!\, Q_\gamma(z) = z^\gamma - \sum_{|\delta| \leqslant k-1} \alpha^{|\delta|}(D^\delta z^\gamma)\, Q_\delta(z), \tag{5}$$

where $\gamma! = \gamma_1! \cdots \gamma_n!$. We may verify that the above polynomials satisfy (as they must by (2))

$$\begin{aligned} \alpha^{|\gamma|}(D^\gamma Q_\beta) &= 0 \quad \text{if} \quad \gamma \neq \beta, \\ &= 1 \quad \text{if} \quad \gamma = \beta \end{aligned} \tag{6}$$

Note that, in case $n = 1$, the formula (5) is only

$$k!\, Q_k(z) = z^k - \sum_{i=0}^{k-1} \alpha^i(D^i z^k)\, Q_i(z). \tag{7}$$

Definition 1.2. Let Ω be an open subset of $\mathbb{C}^n$, $\alpha = (\alpha^0, ..., \alpha^d)$ is called an interpolation sequence (of length d) if $\alpha^i \in H'(\Omega)$ and $\alpha^i(1) = 1$ for $i = 0, ..., d$. We define similarly an infinite interpolation sequence.

The following property follows from the unicity and from (2).

Property 1.3. *The map $f \to L(\alpha, f)$ is a continuous linear projector from $H(\Omega)$ onto $\mathscr{P}_d(\mathbb{C}^n)$ the space of polynomials of degree $\leqslant d$.*

In particular the fact that the map above is a projector will be often used in the following manner: two polynomials $p(z)$ and $q(z)$ of degree $\leqslant d$ are equal if and only if $\alpha^i(D^\beta p) = \alpha^i(D^\beta q)$ for $|\beta| = i$ and $i = 0, 1, ..., d$.

Property 1.4. *Let A be an affine map from $\mathbb{C}^n$ to $\mathbb{C}^m$, Ω open in $\mathbb{C}^n$ and let $\alpha = (\alpha^0, ..., \alpha^d)$ be an interpolation sequence for $H(\Omega)$ then $A * \alpha$ is an interpolation sequence for $H(A(\Omega))$ and if $f \in H(A(\Omega))$:*

$$L(\alpha, f \circ A) = L(A * \alpha, f) \circ A.$$

By definition $A * \alpha$ is $(A * \alpha^0, ..., A * \alpha^d)$ and each $A * \alpha^j$ is the image by A of the functional α^j (i.e., $(A * \alpha^j)(f) = \alpha^j(f \circ A)$).

The usefulness of the above formula is the following: if $f = \int k(\langle \xi, z \rangle)\, d\mu(\xi)$ then to calculate $L(\alpha, f)$, by continuity we have only to interpolate the kernel and by Property 1.4 it is only a one dimensional problem.

Proof of Property 1.4. That $A * \alpha$ is an interpolation sequence for $H(A(\Omega))$ is clear. According to Property 1.3, since the two members of the equality to be proved are polynomials of degree $\leqslant d$, it is enough to verify that for $|\beta| = i$, $i = 0, ..., d$ we have

$$\alpha^i(D^\beta L(\alpha, f \circ A)) = \alpha^i(D^\beta(p \circ A)), \tag{8}$$

where $p(z) = L(A * \alpha, f)$. The calculations are simple but cumbersome. To simplify we just do it in the case $m = 1$, which is extensively used in the sequel. In this case,

$$D^\beta(p \circ A) = (p^{(i)} \circ A)\, \vec{A}^{\beta_1}(e_1) \cdots \vec{A}^{\beta_n}(e_n),$$

where $\vec{A}$ is the linear part of A, (e_i) the canonical basis of $\mathbb{C}^n$. Then

$$\begin{aligned}\alpha^i(D^\beta(p \circ A)) &= \alpha^i(p^{(i)} \circ A)\, \vec{A}^{\beta_1}(e_1) \cdots \vec{A}^{\beta_n}(e_n)\\ &= (A * \alpha^i)(p^{(i)})\, \vec{A}^{\beta_1}(e_1) \cdots \vec{A}^{\beta_n}(e_n)\\ &= (A * \alpha^i)(f^{(i)})\, \vec{A}^{\beta_1}(e_1) \cdots \vec{A}^{\beta_n}(e_n)\\ &= \alpha^i(D^\beta(f \circ A))\\ &= \alpha^i(D^\beta L(\alpha, f \circ A))\end{aligned}$$

and the formula (8) is proved. ∎

2. Some Examples

Whenever specified, Ω is an open subset of $\mathbb{C}^n$.

2.1. *Discrete Interpolation*

We say that a functional $\mu \in H'(\Omega)$ is discrete if of the form

$$\mu(f) = \sum_{i=0}^{\infty} b_i (D^{\beta_i} f)(x_i),$$

where (b_i) is a summable complex sequence, i.e., $\sum_{i=0}^{\infty} |b_i| < \infty$, $X = \{x_i, i = 0, 1, ...\}$ is a relatively compact set in Ω and (β_i) is a sequence of multiindices with bounded length, i.e., $\max\{|\beta_i|, i = 0, 1, ...\} < \infty$. We say that μ is normalized if moreover the sum of all the numbers b_i such that $\beta_i = (0, 0, ..., 0)$ equals 1. This only means that $\mu(1) = 1$. Thus if α^j is a discrete normalized functional for $j = 0, 1, ..., d$ then $\alpha = (\alpha^0, \alpha^1, ..., \alpha^d)$ is an interpolation sequence for $H(\Omega)$. Very particulary if $\alpha^j(f) = f(x_j)$ then we obtain the so called Gontcharrof interpolation; see [11, 12, 8].

2.2. *Kergin Interpolation*

Let $X=(x_0, x_1, ..., x_d)$ be not necessarily distinct points in an open convex subset U of $\mathbb{C}^n$. Define, for $f \in H(U)$ and $i=0, ..., d$,

$$\alpha^i(f) = i! \int_{\Delta^i} f\left(x_0 + \sum_{j=1}^{i} \lambda_j (x_j - x_0)\right) d\lambda, \tag{9}$$

where Δ^i is the standard simplex in $\mathbb{R}^i$,

$$\Delta^i = \left\{(\lambda_j), 0 \leqslant \lambda_j \leqslant 1, j=1, ..., i, \sum_{j=1}^{i} \lambda_j \leqslant 1\right\},$$

and λ is the Lebesgue measure on Δ^i. Then $L(\alpha, f)$ is the so-called Kergin interpolation polynomial of f. It satisfies $L(\alpha, f, x_i) = f(x_i)$ for $i=0, ..., d$ thus in case $n=1$ it is only the classical Lagrange–Hermite polynomial of f at the points $x_0, ..., x_d$ and the numbers $\alpha^i(f^{(i)})$ are the divided differences of f (multiplied by $i!$) with respect to the points $x_0, ..., x_d$.

For further information on Kergin interpolation see [1, 3–5, 17, 18]. Note also that the functionals α^i above may be extended (and hence also the Kergin interpolation) to the case where U is only $\mathbb{C}$-convex (an open set $\Omega \in \mathbb{C}^n$ is $\mathbb{C}$-convex if for each complex line D, $\Omega \cap D$ is simply connected or empty), this is done by Anderson and Passare; see [1].

2.3. *Divided Differences and Other Mean Interpolations*

The remark above leads us, see [8], to define a β-divided difference of a function f of several variables, with respect to the points $x_0, ..., x_d$ by $\alpha^i(D^\beta f)$ with α^i defined by (9) and $|\beta| = i$. Now suppose that (x_i^j) is a triangular array of points and that we know all the β-divided difference of a function with respect to the points $x_i^0, ..., x_i^i$ when $|\beta| = i$ then the corresponding interpolating polynomial is of the type we study.

Gelfond first studied this procedure in $\mathbb{C}$, see [11], then Goodman and Sharma studied it in $\mathbb{C}^n$; see [13]. The results we prove below are proved by them in this case. Some are also first proved by Bloom for Kergin interpolation; see [3].

We see that in this case, the numbers $\alpha^i(f)$ are only means of the function f on a simplex in $\mathbb{C}^n$ with vertices at the points $x_i^0, ..., x_i^i$. It is equally natural to consider means on some other simple convex subsets of $\mathbb{C}^n$. For example, let us consider a family of spheres. For $i=0, 1, ..., d$ we take a point x_i in Ω and a radius r_i such that the euclidean sphere $S(x_i, r_i)$ lies in Ω. Next we consider the usual normalized area measure σ_i on $S(x_i, r_i)$ which clearly defines a functional. Then $\sigma = (\sigma_0, \sigma_1, ..., \sigma_d)$ is an interpolation sequence for $H(\Omega)$.

2.4. *The Gelfond Moment Problem in the Complex Domain*

For $j=0, 1, ..., d$, let γ_j be a piecewise regular curve in the complex plane whose winding number with respect to 0 equals 1 and let

$$\varphi_j(z)=z^{-j-1}+\sum_{m=j+1}^{\infty} a_m^j z^{-m-1}$$

be an holomorphic function in an open neighborhood of γ_j. We suppose that $\Omega(\subset\mathbb{C})$ contains the curves γ_j, $j=0, 1, ..., d$. Gelfond has studied, [11, p. 88] the following moment interpolation problem: given the numbers $\int_{\gamma_j} f(z)\,\varphi_j(z)\,dz$, construct a polynomial $p(z)$ of degree not greater than d such that $\int_{\gamma_j}(f-p)(z)\,\varphi_j(z)\,dz=0$ for $j=0, 1, ..., d$. This is a one variable example of our general procedure. Indeed we just have to take $p(z)=L(\alpha, f, z)$, where $\alpha=(\alpha^0, \alpha^1, ..., \alpha^d)$,

$$\alpha^j(f)=\frac{j!}{2i\pi}\int_{\gamma_j} f(z)\,\psi_j(z)\,dz$$

and

$$\psi_j(z)=\frac{z^{-1}}{j!}+\sum_{m=2}^{\infty}\frac{(m-1)!}{(m+j-1)!}a_{m+j-1}^d z^{-m}.$$

This property is a simple consequence of the following identities:

$$\int_{\gamma_j} f(z)\,\varphi_j(z)\,dz=(-1)^j\int_{\gamma_j} f(z)\,\psi_j^{(j)}(z)\,dz=\int_{\gamma_j} f^{(j)}(z)\,\psi_j(z)\,dz.$$

When applied to this case, the convergence theorems proved below lead to simpler results than those proved in [11].

The discussion above suggests to set a multivariate Gelfond's moment problem in the following manner.

For $j=0, ..., d$ we take in Ω, $\gamma_j=\gamma_{j1}\times\gamma_{j2}\times\cdots\times\gamma_{jn}$ where each γ_{jk} is as above and functions

$$\psi_j(z)=\sum_{k=0}^{n} a_k^j z_k^{-1}+\sum_{|\alpha|>1} a_\alpha^j z^{-\alpha}$$

where $a_1^j, ..., a_n^j$ are chosen in order to ensure that

$$\frac{1}{(2i\pi)^n}\int_{\gamma_{j1}}\cdots\int_{\gamma_{jn}}\psi_j(z)\,dz_1\cdots dz_n=1.$$

Then the problem is to study the polynomial $L(\alpha, f)$ with $\alpha = (\alpha^0, \ldots, \alpha^d)$ and

$$\alpha^j(f) = \frac{1}{(2i\pi)^n} \int_{\gamma_{j1}} \cdots \int_{\gamma_{jn}} f(z)\, \psi_j(z)\, dz_1 \cdots dz_n.$$

For other classical one-dimensional interpolation problems that are of the type we study, we refer to [7]. See also [9, 10] for multivariate interpolation procedures related to those in subsections 2.2 and 2.3.

3. Some Calculations in the One Variable Case

Let $\alpha = (\alpha^d)$ be an infinite interpolation sequence for $H(\Omega)$, Ω being an open subset of the complex plane, the dth basis polynomial $Q_d(z)$ for α is the one defined by (7), it depends only on $\alpha^0, \ldots, \alpha^{d-1}$. We note also $L_d(f) = L(\alpha_d, f)$ where $\alpha_d = (\alpha^0, \ldots, \alpha^d)$ and $\mathscr{R}_d(f, z) = f(z) - L_d(z)$ then:

Lemma 3.1. *For any positive integer d, we have the formula*

$$Q_d(z) = \alpha^0_{\xi_0} \cdots \alpha^{d-1}_{\xi_{d-1}} \left(\int_{\xi_0}^{z} \int_{\xi_1}^{z_1} \cdots \int_{\xi_{d-1}}^{z_{d-1}} dz_1 \cdots dz_d \right). \tag{10}$$

Lemma 3.2. *Suppose Ω convex, then for $d \in \mathbb{N}$, $T \in \mathbb{N}$, $T \leqslant d$ we also have*

$$\mathscr{R}_d^{(T)}(z) = \alpha^T_{\xi_T} \cdots \alpha^d_{\xi_d} \left(\int_{\xi_T}^{z} \int_{\xi_{T+1}}^{z_{T+1}} \cdots \int_{\xi_d}^{z_d} f^{(d+1)}(z_{d+1})\, dz_{T+1} \cdots dz_{d+1} \right). \tag{11}$$

Proofs. In (10) we have $Q_1(z) = z - \alpha^0(\xi_0)$, which is (4), and we verify without difficulty that the polynomials defined by the right side of (10) satisfy the relation (7), hence the formula.

Let us first prove (11) for $T = 0$ and then use induction. The formula (11) is true for $d = 0$. Suppose it is true for $d-1$ then the right term in (11) is equal to

$$\mathscr{R}_{d-1}(z) - \alpha^d(f^{(d)})\, Q_d(z),$$

which is

$$f(z) - L_{d-1}(f, z) - \alpha^d(f^{(d)})\, Q_d(z),$$

that is,

$$f(z) - L_d(f, z),$$

hence the formula is true for d. The general formula follows then by differentiating under the functionals and the lemma is proved. ∎

LEMMA 3.3. *Suppose there exist positive constants* v *and* r *such that for* $d \in \mathbb{N}$ *and* $k \in \mathbb{N}$ *we have* $|\alpha^d(z^k)| \leqslant v r^k$ *then for* $|z| \leqslant \rho$ *we have*

$$|Q_d(z)| \leqslant C(\rho, r, v)(r/v)^d,$$

where $v = \log((1+v)/v)$.

Proof. By (7) and the hypothesis we get

$$|Q_d(z)| \leqslant \frac{|z|^d}{d!} + v \sum_{i=0}^{d-1} \frac{r^{d-i}}{(d-i)!} |Q_i(z)|.$$

Fix $|z| \leqslant \rho$ then $|Q_d(z)| \leqslant a_d$ where (a_d) is the sequence defined by $a_0 = 1$ and

$$a_d = \frac{\rho^d}{d!} + v \sum_{i=0}^{d-1} a_i \frac{r^{d-i}}{(d-i)!}.$$

Next, consider

$$F(x) = \sum_{d \geqslant 0} a_d x^d,$$

by the definition of the coefficients a_d we find

$$(1+v)\, F(x) = \exp(\rho x) + v F(x) \exp rx,$$

hence

$$F(x) = \exp \rho x/(1 + v - v \exp rx).$$

$x_0 = v/r$ is the singularity of F which has the least modulus thus we have

$$|a_d| \leqslant C(\rho, r, v)(r/v)^d$$

and the lemma is proved. ∎

LEMMA 3.4. *Suppose* Ω *convex* $d \geqslant T$ *and let* $M^{(d)} := \sup\{|f^{(d)}(z)|, z \in \Omega\}$, *then we have*

$$|\mathscr{R}_d^{(T)}(z)| \leqslant \frac{M^{(d+1)}}{(d-T+1)!} \int_{\xi_T} \cdots \int_{\xi_d} \left[|z - \xi_T| + \sum_{i=T+1}^{d} |\xi_i - \xi_{i-1}| \right]^{d-T+1} \times |d\tilde{\alpha}^T| \cdots |d\tilde{\alpha}^d|, \tag{12}$$

where $\tilde{\alpha}^i$ *is any measure with compact support in* Ω *which extends* α^i.

Note that since, see (7)

$$Q_{d+1}(z) = \frac{z^{d+1}}{d+1!} - L_d\left(\frac{(\cdot)^{d+1}}{d+1!}, z\right),$$

the estimate (12) *gives also an estimate for* $Q_d^{(T)}(z)$.

Proof. Indeed, the term in brackets in (11) is bounded, see [11, p. 46], by the term in brackets in (12). Note that the convexity of Ω is needed in the proof of the inequality used. ∎

4. Convergence Theorem for Entire Functions: First Case

We note $|z| = \sup |z_i|$, $\langle a, z\rangle = \sum a_i z_i$,

$$M(f, r) = \sup\{|f(z)|, |z| \leqslant r\}.$$

If $\alpha = (\alpha^d)$ is an interpolation sequence for $H(\mathbb{C}^n)$ we note, as in 3, $L_d(f) = L(\alpha_d, f)$ where $\alpha_d = (\alpha^0, ..., \alpha^d)$. The basis polynomials are those defined by (3), (4), (5): a polynomial Q_β depends only on $(\alpha^0, ..., \alpha^{i-1})$ if $|\beta| = i$.

Theorem 4.1. *Let* α *be an infinite interpolation sequence for* $H(\mathbb{C}^n)$ *such that for* $|a| \leqslant 1$, $d \in \mathbb{N}$, $k \in \mathbb{N}$,

$$|\alpha^d(\langle a, z\rangle^k)| \leqslant \nu r^k. \tag{13}$$

If f *is an entire function of exponential type* $< v/r$ *where* $v = \log((1+\nu)/\nu)$, *then* $L_d(f)$ *converges to* f *in* $H(\mathbb{C}^n)$.

Note 1. In fact we will prove the theorem with the following weaker condition on f:

$$\limsup_{t \to \infty} \log[M(f, t)\exp(-tv/r)](\log t)^{-1} < 3/2. \tag{14}$$

Note 2. Since $\alpha^i(1) = 1$, in (13) we have necessarily $\nu \geqslant 1$.

Remark 4.2.

— Equation (13) holds, for some ν and r, when $\alpha = (\alpha^d)$ is an equicontinuous sequence in $H'(\mathbb{C}^n)$ i.e. when there exist a compact set K and a positive number c, both independant of d such that $\alpha^d(f) \leqslant c \max_{z \in K} |f(z)|$ for each d.

— The theorem is of no interest for Kergin interpolation since it is known that for any bounded sequence (a_d) in $\mathbb{C}^n$ and any entire function

f, the Kergin polynomial $K_d(z)$ of f with respect to the points $a_0, a_1, ..., a_d$ converges to f in $H(\mathbb{C}^n)$ as d approaches ∞; see [5].

— We conjecture that for any number $w > v/r$ there exists an infinite interpolation sequence satisfying (13) and an entire function f of exponential type smaller than w such that $L_d(f)$ does not converge to f in $H(\mathbb{C}^n)$.

LEMMA 4.3. *Let l be a linear form on $\mathbb{C}^n$, Q_d be the dth basis polynomial for $l * \alpha$, (e_i) the canonical basis of $\mathbb{C}^n$ then for $d \in \mathbb{N}$, $z \in \mathbb{C}^n$ we have*

$$Q_d(l(z)) = \sum_{|\beta| = d} l^{\beta_1}(e_1) \cdots l^{\beta_n}(e_n)\, Q_\beta(z) \tag{15}$$

(Recall that the Q_β are the basis polynomials for α).

Proof. Since the two sides of (15) are polynomials of degree $\leqslant d$, according to (1.3) it is enough to show that for $i = 0, ..., d$, $|\gamma| = i$,

$$\alpha^i(D^\gamma(Q_d \circ l)) = \sum_{|\beta| = d} l^{\beta_1}(e_1) \cdots l^{\beta_n}(e_n)\, \alpha^i(D^\gamma Q_\beta(z)).$$

This is the same calculation as in the proof of (1.4), so we omit it. ∎

LEMMA 4.4. *With the hypothesis and notations of theorem 4.1, if $|z| \leqslant \rho$, $d \in \mathbb{N}$, $|\beta| = d$ then*

$$|Q_\beta(z)| \leqslant C(\rho, v, r) \left(\frac{r}{v}\right)^d. \tag{16}$$

Proof. Applying lemma 4.3 with $l(z) = l_a(z) = \langle a, z \rangle$ we get

$$Q_d(\langle a, z \rangle) = \sum_{|\beta| = d} a^\beta Q_\beta(z),$$

so that by Cauchy inequalities, see, e.g., [15, Thm. 2.2.7],

$$|Q_\beta(z)| \leqslant \sup\{|Q_d(\langle a, z \rangle)|, |a| \leqslant 1\}.$$

But $Q_d(z)$ is the dth basis polynomial for $l_a * \alpha$ and by (13) the hypothesis of Lemma 3.3 are satisfied, hence

$$|Q_d(\langle a, z \rangle)| \leqslant C(\rho, r, v) \left(\frac{r}{v}\right)^d$$

and the lemma is proved. ∎

Proof of Theorem 4.1.

Step 1. We write $f(z)=\sum a_k z^k=\sum_{s=0}^{\infty} F_s(z)$, $F_s(z)=\sum_{|k|=s} a_k z^k$. Since $f \to L_d(f)$ is a continuous linear projector we have

$$f(z)-L_d(f,z)=\sum_{s=d+1}^{\infty} (F_s(z)-L_d(F_s,z))$$

and

$$L_d(\cdot^k, z)=\sum_{|\gamma|\leqslant d,\, \gamma\leqslant k} \alpha^{|\gamma|}(D^\gamma z^k)\, Q_\gamma(z),$$

but for $|k|=s$,

$$z^k=\sum_{|\gamma|=s,\, \gamma\leqslant k} a^{|\gamma|}(D^\gamma z^k)\, Q_\gamma(z),$$

so that finally we have the error formula

$$f(z)-L_d(f,z)=\sum_{s=d+1}^{\infty}\left[\sum_{|k|=s} a_k \sum_{i=d+1}^{s} \sum_{|\gamma|=i,\, \gamma\leqslant k} \alpha^i(D^\gamma z^k)\, Q_\gamma(z)\right]. \tag{17}$$

In the above formulas $\gamma\leqslant k$ means $\gamma_1\leqslant k_1, ..., \gamma_n\leqslant k_n$; in the sequel we will show that when $|z|\leqslant \rho$ and for some $\varepsilon>0$, the modulus of the term in brackets in (17) is less than or equal to $C(\rho, r, v)\, s^{-(1+\varepsilon)}$. Since $\sum s^{-(1+\varepsilon)}$ is a convergent series (!) the theorem will be proved.

Step 2. For $|\beta|=i$, $\beta\leqslant k$, $|k|=s$ we have

$$|\alpha^i(D^\beta z^k)|\leqslant \frac{k!}{(k-\beta)!}\, r^{s-i} v. \tag{18}$$

In fact we have only to prove that for $|\delta|=d$, $i\in N$,

$$|\alpha^i(z^\delta)|\leqslant v r^d,$$

but for $|a|\leqslant 1$,

$$v r^d \geqslant \sup_{|a|\leqslant 1} |\alpha^i(\langle a, z\rangle)^d| = \sup_{|a|\leqslant 1} \left|\sum_{|\beta|=d} a^\beta \alpha^i(z^\beta)\right| \geqslant |\alpha^i(z^\delta)|.$$

The first inequality above is true by hypothesis and the second by Cauchy inequalities, hence (18) is proved.

Step 3. Now using (18) and (16) we find that, when $|z|\leqslant \rho$, the term in brackets in (17) is less than or equal to

$$C(\rho, r, v)\sum_{|k|=s} |a_k| \sum_{i=d+1}^{s} \sum_{|\gamma|=i,\, \gamma\leqslant k} \frac{k!}{(k-s)!}\, r^{s-i}\left(\frac{r}{v}\right)^i.$$

Since $v \geqslant 1$, $v \leqslant 1$ so that the above term is still not greater than

$$C(\rho, r, v) \sum_{|k|=s} |a_k| \frac{r^s}{v^s} k!. \tag{19}$$

Now by the growth hypothesis (14) and the Cauchy inequalities, for some $\varepsilon > 0$ and t large enough we have

$$|a_k| \leqslant \frac{t^{(3/2)-\varepsilon}}{t^s} \exp\left(\frac{tv}{r}\right).$$

Next we take $t = t(s) = sr/v$ in the above estimate and plug it in (19).

If we remark that $\sum_{|k|=s} (k!/s!) = O(1)$ we may use the Stirling formula to find that (19) is less than or equal to

$$C(\rho, r, v)\, s^{-(1+\varepsilon)},$$

so that according to the first step, the theorem is proved. ∎

If S is a compact convex subset of $\mathbb{C}^n$, we define

$$H_S(\xi) = \sup\{\operatorname{Re}(\langle \xi, z\rangle), z \in S\}.$$

COROLLARY 4.5. *Let S be a compact convex subset of $\mathbb{C}^n$, α an interpolation sequence for $H(\mathbb{C}^n)$ such that for $a \in S$, $d \in \mathbb{N}$, $k \in \mathbb{N}$,*

$$|\alpha^d(\langle a, z\rangle)^k| \leqslant r^k, \tag{20}$$

where r is strictly less than $\log 2$. *Suppose that f is an entire function such that for any $\varepsilon \geqslant 0$ there exists $C(\varepsilon) > 0$,*

$$|f(\xi)| < C(\varepsilon) \exp(H_S(\xi)) + \varepsilon\, |\xi|) \qquad (\xi \in \mathbb{C}^n), \tag{21}$$

then $L_d(f)$ converges to f in $H(\mathbb{C}^n)$.

Remark 4.6.

— If the right side in (20) is replaced more generaly by vr^k then the corollary is true if r is strictly less than $\log((1+v)/v)$.

— The condition (21) says only that f is the Fourier–Borel transform of an analytic functional carried by S.

Proof. Choose K an open convex set close enough to S to satisfy for $a \in K$, d, $k \in \mathbb{N}$,

$$|\alpha^d(\langle a, z\rangle^k)| \leqslant \delta^k, \tag{22}$$

where δ is still strictly less than $\log 2$. By (21) there exists a bounded measure with compact support in K, see [19], such that

$$f(z)=\int_K \exp(\langle \xi, z\rangle)\, d\mu(\xi) \qquad (z\in\mathbb{C}^n). \tag{23}$$

Now we interpolate the kernel, using Property 1.4. By (22) and Theorem 4.1 in the one dimensional case the interpolated kernel converges uniformly in z to the kernel, but a simple look at the constants which appear in the proof of Theorem 4.1 shows that the convergence is also uniform for $\xi\in K$. ▌

EXAMPLE 4.7. With the hypothesis of Theorem 4.1 or Corollary 4.5, if $\alpha^{|\beta|}(D^\beta f)=0$ for $|\beta|=d$ then f must be a polynomial of degree less than d.

Remark 4.8. The constant $\log 2$ is not optimum for Gontcharoff interpolation but better constants in the one variable case, Ref. [6] will give better constants in the several variables case via (23).

5. CONVERGENCE THEOREM FOR ENTIRE FUNCTIONS: SECOND CASE

We now use the euclidean norm $\|z\|^2=\sum |z_i^2|$, define

$$M(f,r)=\sup\{|f(z)|,\ \|z\|=r\}$$

and work with an interpolation sequence α for $H(\mathbb{C}^n)$ satisfying the condition (★) below:

There exist v and for each d a compact set K_d such that

$$|\alpha^d(f)|\leqslant v\,\|f\|_{K_d}:=v\max_{z\in K_d}|f(z)|. \tag{★}$$

The point is that v does not depend on d; note also that K_d is not unique, we have to choose one (roughly speaking a small one). We suppose that $\bigcup_{d\geqslant 0}K_d$ is unbounded otherwise we are reduced to the first case. To measure the unboundedness of the sequence K_d we introduce the following objects:

1. for $\rho\geqslant 0$, $D_0(\rho)=\sup\{\|z-\xi_0\|,\ \|z\|\leqslant\rho,\ \xi_0\in K_0\}$,
2. for $d\geqslant 1$, $D_d=\operatorname{diam}(K_d\cup K_{d-1})$,
3. for $d\geqslant 1$, $\tau_d=\sum_{i=1}^d D_i$,
4. the function $N(r)$, $r\geqslant 0$ is defined by

$$N(r)=k \quad \text{if} \quad \tau_k\leqslant r \quad \text{and} \quad \tau_{k+1}>r.$$

Since $\bigcup K_d$ is unbounded, we have $\lim_{n\to\infty} \tau_d = \infty$ so that the function N is well defined.

In case of Kergin interpolation, see Subsection 2.2, the function $N(r)$ is the classical counting function of the sequence of interpolation that is $N(r)$ is the number of nodes in the ball $\{\|z\| \leqslant r\}$.

Finally note that for the function N the following property always holds:

$$(\tau_d \leqslant r \Rightarrow d \leqslant N(r)) \qquad \text{and} \qquad (\tau_d > r \Rightarrow d > N(r)). \tag{24}$$

With these notations and definitions we have:

THEOREM 5.1. *Let α be an interpolation sequence such that ($\bigstar$) holds. Let f be an entire function whose growth satisfies*

$$\log M(f, r) \leqslant \lambda N(\theta r)$$

for r large enough and, with $0 < \lambda < \log((1-\theta)/\nu\theta)$ and $0 < \theta < 1/(\nu + 1)$, ν being the constant that appears in ($\bigstar$). Then $L_d(f)$ converges to f in $H(\mathbb{C}^n)$.

Proof. For a fixed $\rho > 0$, we are going to show that $L_d(f)$ converges uniformly to f on $\{\|z\| \leqslant \rho\}$.

Take $r(d)$ and $R(d)$ two sequences (to be specified later) such that

(H1) $R(d) > r(d) > \rho$

(H2) $\alpha_d = (\alpha^0, ..., \alpha^d)$ is an interpolation sequence for $H(\{\|z\| < R(d)\})$.

To get (H2) it is enough that the K_i of ($\bigstar$) lie in $\{\|z\| < R(d)\}$ for $i = 0, ..., d$.

Recall that the Cauchy representation formula, see [2], gives for $\|z\| \leqslant r(d)$:

$$f(z) = \frac{R(d)(n-1)!}{2\pi^n} \int_{S(R(d))} \frac{f(\xi)}{(R^2(d) - \langle z, \bar{\xi}\rangle)^n} \, d\sigma(\xi),$$

where $d\sigma(\xi)$ is the area measure on the sphere $S(R(d)) = \{\|z\| = R(d)\}$.

Denoting by $g_d(u)$ the function $1/(R^2(d) - u)^n$ and using (H2), 1.3, and 1.4 we get

$$f(z) - L_d(f, z) = \frac{R(d)(n-1)!}{2\pi^n} \int_{S(R(d))} [(g_d - L_d(l_{\bar{\xi}} * \alpha_d, g_d))(\langle z, \bar{\xi}\rangle)] \, f(\xi) \, d\sigma(\xi), \tag{25}$$

where $l_{\bar{\xi}} = \langle z, \bar{\xi}\rangle$.

Now choose η such that $\theta < 1/\eta < 1/(1+v)$ and $\log((\eta-1)/v) > \lambda$ (this is possible since $\log((1-\theta)/v\theta) > \lambda$) and take

$$R(d) = \eta r(d), \qquad r(d) = D_0(\rho) + \tau_d.$$

The estimate of Lemma 3.4 may be used to bound the term in brackets in (25). Taking into account that $\|\xi\| = R(d)$, we get for $\|z\| \leqslant \rho$:

$$\begin{aligned} |f(z) - L_d(f, z)| &\leqslant \sup\{|f(z) - L_d(f, z)|, \|z\| = r(d)\} \\ &\leqslant CM(f, R(d)) \frac{n(n+1)\cdots(n+d)}{(d+1)!} \\ &\quad \times \frac{(R(d)\, r(d)\, v)^{d+1}\, R(d)^{2n}}{(R^2(d) - R(d)\, r(d))^{n+d+1}} \qquad (26) \\ &\leqslant CM(f, R(d))\, d^{n-1} \left(\frac{v}{\eta-1}\right)^d \\ &\leqslant C \exp((\lambda N(\theta\eta r(d)))\, d^{n-1} \left(\frac{v}{\eta-1}\right)^d. \end{aligned}$$

We have used that

$$\frac{n(n+1)\cdots(n+d)}{d+1\,!} \sim \frac{d^{n-1}}{n-1\,!}$$

and the hypothesis on the growth of f.

For d large enough we have $\theta\eta(D_0(\rho) + \tau_d) < \tau_d$ since $\theta\eta < 1$ and $\lim_{n\to\infty} \tau_d = \infty$ hence $N(\theta\eta r(d)) \leqslant d$.

Finally, for $\|z\| \leqslant \rho$ we have

$$|f(z) - L_d(f, z)| \leqslant C \exp\left[\left(\lambda + \log\left(\frac{v}{\eta-1}\right)\right) d + (n-1)\log d\right]$$

but $-\log(v/(\eta-1)) > \lambda$ hence the term in brackets in the formula above tends to $-\infty$ as d tends to ∞ so that $L_d(f, z)$ converges uniformly to f on $\{\|z\| \leqslant \rho\}$ and the theorem is proved. ∎

EXAMPLE 5.2. Let $u \in \mathbb{C}^n$, with $\|u\| = 1$, $x_d = du$, $\alpha = (\delta_{x_d})$. In this case $N(r) = E(r)$ where E is the integral part of r. If f is an entire function satisfying $D^\beta f(du) = 0$, $|\beta| \geqslant 0$, and

$$M(f, R) \leqslant \exp(\lambda E(\theta R))$$

with θ and λ as in Theorem 5.1 then $f = 0$. This result is to be compared with the one in [14].

Remark 5.3. Theorem 5.1 may not by improved without diminishing the generality, indeed the hypothesis on θ is optimum in case of Gontcharoff interpolation; see [16].

COROLLARY 5.4 (to the proof). *Let f be a non zero entire function of order $\leqslant \mu$; if the polynomial $L_d(f)$ is zero for each d, then it is necessary that the function $N(r)$ is of order $\leqslant \mu$.*

Recall that a positive function h defined for $r \geqslant 0$ is said to be of order μ if $h(r) = O(r^{\mu+\varepsilon})$ for each positive ε but for no negative ε; an entire function f is of order μ if the function $\log M(f, r)$ is of order μ.

Proof. Without loss of generality we may suppose that $f(0) \neq 0$. Fix $R > 0$, take $d \in \mathbb{N}$ such that $\tau_{d+1} > R \geqslant \tau_d$ and write the formula (26) taking $R(d) = (\nu+2)\, r(d)$ and $r(d) = D_0(0) + \tau_d$, $z = 0$. This is possible since the hypothesis (H2) in the proof of Theorem 5.1 still holds here. We get after some calculations and since $L_d(f, 0) = 0$,

$$|f(0)| \leqslant CM(f, (\nu+2)\, r(d))\, d^{n-1} \left(\frac{\nu}{\nu+1}\right)^{d+1},$$

then

$$(d+1) \log\left(\frac{\nu+1}{\nu}\right) - (n-1) \log d \leqslant C + \log M(f, (\nu+2)\, r(d)).$$

For d large enough, that is, for R large enough we will have

$$\frac{d+1}{2} \log\left(\frac{\nu+1}{\nu}\right) \leqslant C + \log M(f, (\nu+2)\, r(d)),$$

but $3r(d) \leqslant 3R + 3D_0(0)$ and $d+1 = N(\tau_{d+1}) > N(R)$, hence

$$N(R)\frac{1}{2} \log\left(\frac{\nu+1}{\nu}\right) \leqslant C + \log M(f, (\nu+2)\, R).$$

Since the function on the right side above is of order $\leqslant \mu$ it is the same for $N(R)$. ∎

COROLLARY 5.5 (to the Proof of 5.1). *Let f be an entire function of order $\leqslant \mu$. Suppose that the function $N(R)$ is of order $> \mu$; then there exists a sequence d_k, $k \in \mathbb{N}$ such that $L_{d_k}(f)$ converges to f in $H(\mathbb{C}^n)$. The sequence d_k depends on N, that is on α but not on f.*

Proof. Since $N(R)$ is of order $> \mu$, we may find a sequence R_k and $\mu' > \mu$ such that $\lim_{k \to \infty} R_k = \infty$ and $N(R_k) \geqslant R_k^{\mu'}$.

For each k, choose $d_k \in \mathbb{N}$ such that $\tau_{d_k+1} > R_k \geqslant \tau_{d_k}$. Hence for each k,

$$d_k = N(\tau_{d_k}) = N(R_k) \geqslant R_k^{\mu'} \geqslant \tau_{d_k}^{\mu'}. \tag{27}$$

We are going to show that $L_{d_k}(f)$ converges to f in $H(\mathbb{C}^n)$.

Fix $\rho > 0$, define $r(d) = D_0(\rho) + \tau_d$ and $R(d) = (\nu+2)\, r(d)$ then the formula (26) gives for $\|z\| \leqslant \rho$

$$|f(z) - L_{d_k}(f, z)| \leqslant CM(f, R(d_k))\, d_k^{n-1} \left(\frac{\nu}{\nu+1}\right)^{d_k}.$$

Now we take μ'' and μ''' such that $\mu < \mu'' < \mu''' < \mu'$, since f is of order $\leqslant \mu$,

$$M(f, R(d_k)) \leqslant C(\mu'') \exp((\nu+2)^{\mu''} (r(d_k))^{\mu''}).$$

On the other hand for k large enough,

$$(\tau_{d_k})^{\mu'''} \geqslant (D_0(\rho) + \tau_{d_k})^{\mu''},$$

and then by (27),

$$(d_k)^{\mu'''/\mu'} \geqslant (r(d_k))^{\mu''},$$

so that for $\|z\| \leqslant \rho$,

$$|f(z) - L_{d_k}(f, z)| \leqslant C \exp\left[(\nu+2)^{\mu''} (d_k)^{\mu'''/\mu'} + (n-1) \times \log d_k - d_k \log\left(\frac{\nu+1}{\nu}\right)\right].$$

The corollary is proved since the term inside the brackets in the above formula tends to $-\infty$ when k tends to ∞. ∎

6. A Convergence Theorem for Functions Analytic in a Neighborhood of the Origin

Let $R > 0$, α an interpolation sequence for $H(\{\|z\| < R\})$, we consider the following three conditions:

(★) For each $d \in \mathbb{N}$ there exists K_d such that

$$|\alpha^d(f)| \leqslant \|f\|_{K_d}.$$

($\bigstar\bigstar$) For each $\varepsilon \geqslant 0$, there exists $d(\varepsilon)$ such that $d > d(\varepsilon)$ implies that K_d lies in the euclidean ball with radius ε and center the origin.

($\bigstar\bigstar\bigstar$) $\sum_{d=1}^{\infty} \mathrm{Diam}(K_d \cup K_{d-1}) < \infty$.

These conditions are very strong, roughly speaking they mean that $L_d(f)$ is quite "near" the Taylor polynomials of f.

THEOREM 6.1. *Let α be an interpolation sequence for $H(\{\|r\| < R\})$ such that the conditions ($\bigstar$), ($\bigstar\bigstar$), ($\bigstar\bigstar\bigstar$) hold. If f is an analytic function in $\{\|z\| < R\}$ then $L_d(f)$ converges to f in $H(\{\|z\| < R\})$.*

Proof of Theorem 6.1. Let $0 < R_1 < R$. We must show that $L_d(f)$ tends to f uniformly on $\{|z| \leqslant R_1\}$. To do this, via the Cauchy representation formula, it is enough to prove that $L_d(\mathscr{C}_\xi, z)$ converges uniformly for $|\xi| = R_2$ and $\|z\| \leqslant R_1$ to $\mathscr{C}_\xi(z)$ where R_2 is any fixed number between R_1 and R, and by definition

$$\mathscr{C}_\xi(z) = \frac{1}{(R_2^2 - \langle \bar{\xi}, z \rangle)^n} = \mathscr{C}(\langle \bar{\xi}, z \rangle).$$

To simplify we define for $\xi \in \mathbb{C}^n$,

$$\alpha^{d,\xi} = l_{\bar{\xi}} * \alpha^d,$$

$$\alpha_\xi = l_{\bar{\xi}} * \alpha,$$

where, as in part 5, $l_{\bar{\xi}}(z) = \langle z, \bar{\xi} \rangle$, $Q_{d,\xi}(z)$, the dth basis polynomial for the interpolation sequence α_ξ, finally $L_{d,\xi}(h)$ is the dth interpolating polynomial of h for the one dimensional interpolation sequence α_ξ. Because of Property 1.4, we have always

$$L_d(\mathscr{C}_\xi, z) = L_{d,\xi}(\mathscr{C}, \langle \bar{\xi}, z \rangle). \tag{28}$$

Now, first we fix $\varepsilon > 0$ small enough to verify

$$3\varepsilon R_2 < R_2^2 - \varepsilon R_2 \tag{29}$$

and

$$R_1 R_2 + 2\varepsilon R_2 < R_2^2 - \varepsilon R_2. \tag{30}$$

Second, we fix an integer T large enough to verify

$$\sum_{d=T}^{\infty} \mathrm{Diam}(K_d \cup K_{d-1}) \leqslant \varepsilon \tag{31}$$

and

$$(d \geqslant T) \Rightarrow K_d \subset \{\|z\| \leqslant \varepsilon\}. \tag{32}$$

Step 1. We show that $L^{(T)}_{d,\xi}(\mathscr{C}, u)$ converges to $\mathscr{C}^{(T)}(u)$ uniformly for $\|\xi\| = R_2$ and $|u| \leqslant \varepsilon R_2$. By (31) and Lemma 3.4,

$$|\mathscr{C}^{T)}(u) - L^{(T)}_{d,\xi}(\mathscr{C}, u)| \leqslant M^{(d+1)}(\varepsilon R_2)\,\frac{(3\varepsilon R_2)^{d-T+1}}{(d-T+1)!}, \tag{33}$$

where

$$M^{(d+1)}(\varepsilon R_2) \leqslant \sup\{|\mathscr{C}^{(d+1)}(u)|,\ |u| \leqslant \varepsilon R_2\} = \frac{n(n+1)\cdots(n+d)}{(R_2^2 - \varepsilon R_2)^{d+n+1}},$$

hence the left term in (33) is bounded by

$$C(T, \varepsilon)\,\frac{n(n+1)\cdots(n+d)}{(d-T+1)!}\left(\frac{3\varepsilon R_2}{R_2^2 - \varepsilon R_2}\right)^{d+1},$$

which tends to 0 as d tends to ∞, ε being well chosen; see (29). The first step is proved.

Step 2. We prove that $L^{(T)}_{d,\xi}(\mathscr{C}, u)$ converges uniformly for $|u| \leqslant R_1 R_2$. It is enough to show that the series of general term

$$L^{(T)}_{d+1,\xi}(\mathscr{C}, u) - L^{(T)}_{d,\xi}(\mathscr{C}, u)$$

is uniformly convergent for $|u| \leqslant R_1 R_2$. But the above term is also equal to

$$\alpha^{d+1,\xi}(\mathscr{C}^{(d+1)})\, Q^{(T)}_{d+1,\xi}(u). \tag{34}$$

For $d \geqslant T$, see (★),

$$\alpha^{d+1,\xi}(\mathscr{C}^{(d+1)}) \leqslant M^{(d+1)}(\varepsilon R_2),$$

and by Lemma 3.4, if $|u| \leqslant R_1 R_2$,

$$|Q^{(T)}_{d+1,\xi}(u)| \leqslant \frac{(R_1 R_2 + 2\varepsilon R_2)^{d-T+1}}{(d-T+1)!}.$$

By the two estimates above we deduce that (34) is bounded (in absolute value) by

$$C(T, \varepsilon)\,\frac{n(n+1)\cdots(n+d)}{(d-T+1)!}\left(\frac{R_1 R_2 + 2\varepsilon R_2}{R_2^2 - \varepsilon R_2}\right)^{d+1}, \tag{35}$$

hence, because of (30), the second step is proved.

Then, because of the first step we conclude that $L_{d,\xi}^{(T)}(u)$ converges to $\mathscr{C}^{(T)}(u)$ uniformly for $|u| \leqslant R_1 R_2$ and $\|\xi\| = R_2$.

Step 3. Integrating T times on the segment $[0, u]$ we find that

$$L_{d,\xi}(\mathscr{C}, u) + \sum_{i=0}^{T-1} L_{d,\xi}^{(i)}(\mathscr{C}, 0) \frac{u^i}{i!} \tag{36}$$

converges uniformly for $\|\xi\| = R_2$ and $|u| \leqslant R_1 R_2$ to

$$\mathscr{C}(u) + \sum_{i=0}^{T-1} \mathscr{C}^i(o) \frac{u^i}{i!}. \tag{37}$$

Differentiating $T-1$ times (36) and (37) and applying then $\alpha^{T-1,\xi}$ we get that

$$L_{d,\xi}^{(T-1)}(\mathscr{C}, 0)$$

converges to $\mathscr{C}^{(T-1)}(0)$, uniformly in ξ, so that

$$L_{d,\xi}(\mathscr{C}, u) + \sum_{i=0}^{T-2} L_{d,\xi}^{(i)}(\mathscr{C}, 0) \frac{u^i}{i!}$$

converges to

$$\mathscr{C}(u) + \sum_{i=0}^{T-2} \mathscr{C}^{(i)}(0) \frac{u^i}{i!}.$$

Now we do the same with $\alpha^{T-2,\xi}$, then $\alpha^{T-3,\xi}$ etc.... Finally we conclude that $L_{d,\xi}(\mathscr{C}, u)$ converges to $\mathscr{C}(u)$ uniformly for $\|\xi\| = R_2$ and $|u| \leqslant R_1 R_2$, taking $u = \langle z, \bar{\xi} \rangle$, $\|z\| \leqslant R_1$ we get that $L_{d,\xi}(\mathscr{C}, \langle \mathscr{C}, \langle z, \bar{\xi} \rangle)$ converges to $\mathscr{C}_\xi(z)$ uniformly for $\|\xi\| = R_2$ and $\|z\| \leqslant R_1$. The theorem is proved. ∎

Of course we may replace the origin by any other point to get a more general theorem.

EXAMPLE 6.2. If f is a non zero entire function such that for each β, with $|\beta| = d$,

$$D^\beta(f)(x_d) = 0,$$

then the series $\sum_{d=1}^{\infty} \|x_{d+1} - x_d\|$ necessarily diverges.

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